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On the Classical Rod that is accreting in Cross-Sectional Area and Length subjected to Longitudinal Vibrations

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Abstract: In this paper a classical rod that is slender and cylindrical in shape, subjected to free longitudinal vibrations, is considered. The rod is assumed to be accreting in length as well as in cross-sectional area. The rod's growth is proportional to time. The vibrating rod is configured such that it is fixed at the left end and free at the other end. A boundary-value problem is then derived consistent with the dynamics of the rod as well as the boundary conditions as per the configurations of the rod. The change of variables is introduced so that the derived partial differential equation could be solved with much ease. This derived equation is further simplified by introducing small parameters to effect the slow growth of the rod. The equation is then solved using the numerical method, the Galerkin- Kantorovich method. It is shown in the solution of this differential equation that there is an increase in the amplitude of vibration at any given mode of vibration. It is further shown that the amplitude of vibration decreases as we move from one mode of vibration to the next one.

Keywords*: Accreting rod, amplitude of vibration, free longitudinal vibrations, Garlekin-Kantorovich method, resonance*

I. INTRODUCTION

The study of a longitudinally vibrating rod that is accreting in one or several dimensions is but just one example of the many arising from the theory of growing structures. This theory of accreting bodies is a new and fast developing branch of analytical mechanics [1]. It is a theory based on partial differential equations which have since become the most important tools in the modelling and formulation of the fundamental laws of nature and in the mathematical analysis of a variety of problems in applied mathematics and engineering science. Shatalov et al in [1] has dealt in great details the behavioural patterns of the longitudinally vibrating rod that is increasing in length, while subjected to free vibrations. This work was extended further by Shatalov et al in [2] when he worked on the accreting rod subjected to damped and forced longitudinal vibrations. In both these works some amazing and interesting results were obtained and discussed.

In this paper we consider a classical rod that is accreting in both the length and the cross-sectional area. The rod, assumed to be of unit length, is subjected to free longitudinal vibrations. The rod is fixed at the left end and free at the other end. The problem arising and the dynamics of this vibrating rod are modelled and described by the partial differential equation of the wave form. The derived partial differential equation is solved numerically using the Galerkin-Kantorovich method [3]. With this method the governing equation is transformed into an infinite system of ordinary differential equations [4]. The system of ordinary differential equations is conveniently truncated to a system of five ordinary differential equations. The solutions of these equations are plotted and then analysed qualitatively for the behavioural patterns of such a vibrating rod.
It is shown in these graphical solutions that the amplitudes of vibration increase with time at any mode of vibration, thus signalling a resonance phenomenon. It is further shown that there is a marked decrease in amplitude of vibration as we move from one mode of vibration to the next.

II. FORMULATION OF THE EQUATION OF THE ACCRETING ROD

The vibrating rod considered here is assumed to be of unit length, and its physical parameters such as the modulus of elasticity E and the mass density, ρ, are constant. The equation of the longitudinal vibration describing the longitudinal displacement $u = u(t, x)$ is given:

$$\frac{\partial}{\partial t}\left(\rho\, A(t)\frac{\partial u}{\partial t}\right) - E\, A(t)\frac{\partial^2 u}{\partial x^2} = 0. \qquad (1)$$

The radius of the cross-sectional area of the rod is assumed variable, a function of time t. The radial growth therefore effects the cross-sectional area growth of the rod with time. The radial growth is defined by

$$r(t) = r_0 + \varepsilon\, \eta\, t^{\frac{\alpha}{2}} \qquad (2)$$

where $\varepsilon\ and\ \eta$ are small parameters to effect the slow rate of growth of the rod, and α is a small parameter describing the general case of the radial growth. In this paper we only consider the case for $\alpha = 1$, for convenience. With the cross-sectional area of the rod defined by

$$A = A(t) = \pi\left(r_0 + \varepsilon\, \eta\, \sqrt{t}\right)^2 \qquad (3)$$

equation (1) can now be written as:

$$\frac{\partial^2 u}{\partial t^2}+\frac{\varepsilon\,\eta}{(r_0+\varepsilon\,\eta\,\sqrt{t})\sqrt{t}}\frac{\partial u}{\partial t}-c^2\frac{\partial^2 u}{\partial x^2}=0 \qquad (4)$$

where c is the speed of the longitudinal propagation of waves. It is further assumed that the rod accretes in length at the same time it grows in cross-sectional area. The growth in length, at any time t, is defined by

$$x=1+\ \varepsilon\,f(t). \qquad (5)$$

With this growth, the boundary conditions are still defined consistent with the configuration of the rod as

$$x=0:\quad u(t,x)=0$$

$$x=1+\ \varepsilon\,f(t):\quad u'(t,x)=0. \qquad (6)$$

Introducing a change of variables such that

$$t=\tau\ and\ x=y.[1+\ \varepsilon\,f(t)] \qquad (7)$$

the boundary conditions in these new variables are

$$y=0:\quad \tilde{u}(\tau,y)=0$$

$$y=1:\quad \frac{\partial\tilde{u}(\tau,y)}{\partial y}=0. \qquad (8)$$

In these new variables equation (4), using the *Mathematica*® 7.0 software, is transformed into the following partial differential equation

$$\frac{\partial^2\tilde{u}}{\partial\tau^2}+\frac{2y\varepsilon^2 f'(\tau)}{(1+\varepsilon\,\tau)^2}\frac{\partial\tilde{u}}{\partial y}-\frac{y\varepsilon f''(\tau)}{(1+\varepsilon\,\tau)}\frac{\partial\tilde{u}}{\partial y}-\frac{c^2}{(1+\varepsilon\,\tau)^2}\frac{\partial^2\tilde{u}}{\partial y^2}+\frac{\varepsilon\,\eta\left(-\frac{y\,\varepsilon\,f'(\tau)}{(1+\varepsilon\,\tau)}\frac{\partial\tilde{u}}{\partial y}+\frac{\partial\tilde{u}}{\partial\tau}\right)}{(r_0+\varepsilon\,\eta\,\sqrt{\tau})\sqrt{\tau}}-\frac{y\,\varepsilon\,f'(\tau)}{(1+\varepsilon\,\tau)}\frac{\partial^2\tilde{u}}{\partial\tau\partial y}$$

$$\frac{y\,\varepsilon\,f'(\tau)\left(-\frac{y\,\varepsilon\,f'(\tau)\partial^2\tilde{u}}{(1+\varepsilon\,\tau)\,\partial y^2}+\frac{\partial^2\tilde{u}}{\partial\tau\partial y}\right)}{(1+\varepsilon\,\tau)}=0. \qquad (9)$$

Having assumed that $\varepsilon\ and\ \eta$ are small parameters, it is conveniently possible to neglect terms of $O(\varepsilon^2)$ and/or $O(\varepsilon\,\eta)$ in the numerators of terms in this partial differential equation (9). With a further assumption that the rod is growing linearly as

$$f(\tau)=\tau, \qquad (10)$$

equation (9) can be simplified to the form:

$$\frac{\partial^2\tilde{u}}{\partial\tau^2}+\frac{\varepsilon\,\eta}{(r_0+\varepsilon\,\eta\,\sqrt{\tau})\sqrt{\tau}}\frac{\partial\tilde{u}}{\partial\tau}-\frac{2\,y\,\varepsilon}{(1+\varepsilon\,\tau)}\frac{\partial^2\tilde{u}}{\partial\tau\partial y}-\frac{c^2}{(1+\varepsilon\,\tau)^2}\frac{\partial^2\tilde{u}}{\partial y^2}=0. \qquad (11)$$

The numerical solution of equation (11) is obtained by using the Galerkin-Kantorovich method [3]. In this method a sequence of linearly independent functions of the form

$$\tilde{u}(y)=\tilde{u}=sin\left[\frac{(2k+1)\pi}{2}y\right], for\ k=0,1,2,3,\ldots \qquad 12)$$

called the basis function, is chosen such that it satisfies the boundary conditions as in equation (8). The following linear combination of functions is then chosen as an approximate solution of the partial differential equation in (11):

$$\tilde{u}(\tau,y)=\sum_{m=1}^{\infty}C_m(\tau)\,sin\left[\frac{(2m-1)\pi}{2}y\right] \qquad (13)$$

with the unknown coefficients $C_m(\tau)$ still to be determined in the process of solving this boundary-value problem. In solving this problem, equation (13) is substituted into equation (11), the result of which is multiplied by equation (12). The result hereof is then integrated over the interval $\{0\le y\le 1\}$. The following system of coupled ordinary differential equations is obtained using *Mathematica*® 7.0 software:

$$\frac{d^2C_1}{d\tau^2}+q_1\frac{dC_1}{d\tau}+q_2C_1+\frac{\varepsilon}{60(1+\varepsilon\tau)}\left[270\frac{dC_2}{d\tau}-250\frac{dC_3}{d\tau}+245\frac{dC_4}{d\tau}-243\frac{dC_5}{d\tau}\right]=0$$
$$\frac{d^2C_2}{d\tau^2}+q_1\frac{dC_2}{d\tau}+9q_2C_2+\frac{\varepsilon}{20(1+\varepsilon\tau)}\left[-10\frac{dC_1}{d\tau}+125\frac{dC_3}{d\tau}-98\frac{dC_4}{d\tau}+90\frac{dC_5}{d\tau}\right]=0$$
$$\frac{d^2C_3}{d\tau^2}+q_1\frac{dC_3}{d\tau}+25q_2C_3+\frac{\varepsilon}{84(1+\varepsilon\tau)}\left[14\frac{dC_1}{d\tau}-189\frac{dC_2}{d\tau}+686\frac{dC_4}{d\tau}-486\frac{dC_5}{d\tau}\right]=0$$
$$\frac{d^2C_4}{d\tau^2}+q_1\frac{dC_4}{d\tau}+49q_2C_4+\frac{\varepsilon}{120(1+\varepsilon\tau)}\left[-10\frac{dC_1}{d\tau}+180\frac{dC_2}{d\tau}-500\frac{dC_3}{d\tau}+1215\frac{dC_5}{d\tau}\right]=0$$
$$\frac{d^2C_5}{d\tau^2}+q_1\frac{dC_5}{d\tau}+81q_2C_5+\frac{\varepsilon}{28(1+\varepsilon\tau)}\left[14\frac{dC_1}{d\tau}-140\frac{dC_2}{d\tau}+500\frac{dC_3}{d\tau}-1715\frac{dC_4}{d\tau}\right]=0$$

(14)

where

$$q_1=\varepsilon\left[\frac{-1}{(1+\varepsilon\tau)}+\frac{\eta}{(r_0+\varepsilon\,\eta\,\sqrt{\tau})\sqrt{\tau}}\right]\ and\ q_2=\frac{c^2\pi^2}{4(1+\varepsilon\tau)^2} \qquad (15)$$

III. NUMERICAL ANALYSIS

The system of equations (14) is numerically solved using the *Mathematica*® 7.0 software. The following values were assumed for the constants in the derived differential equation: $c = 1$; $\varepsilon = 0.05$; $r_0 = 0.1$ and $\eta = 0.5$.The solutions to these differential equations are given by the corresponding graphs below:

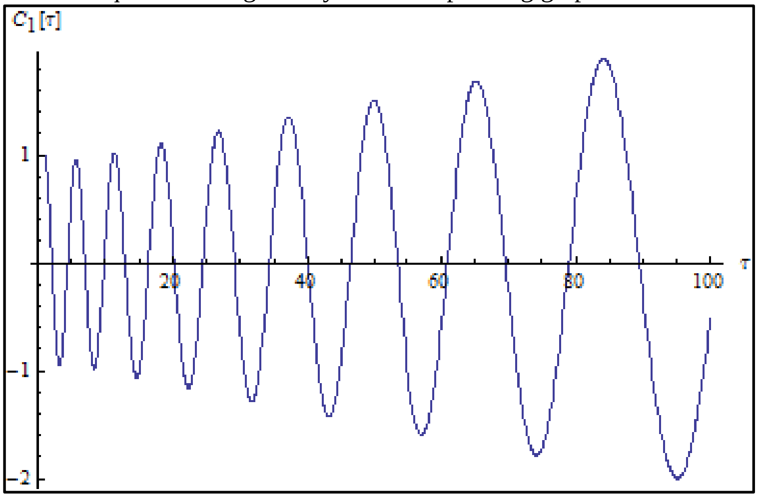

Fig. 1 Resonance at First Mode

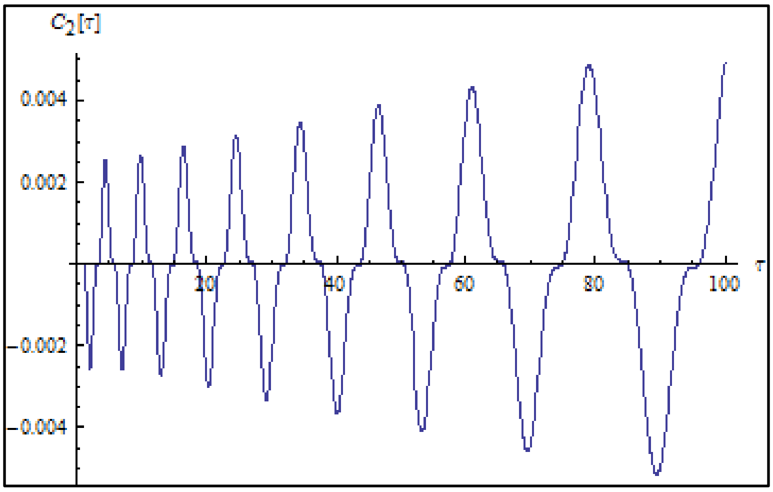

Fig. 2 Resonance at Second Mode

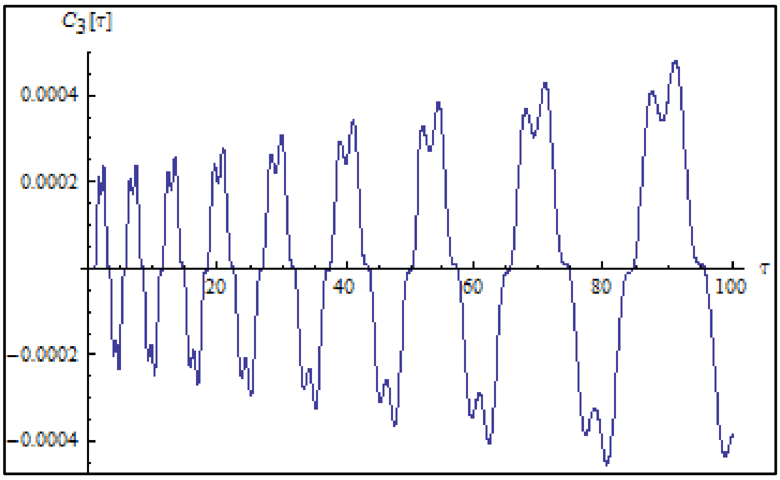

Fig. 3 Resonance at Third Mode

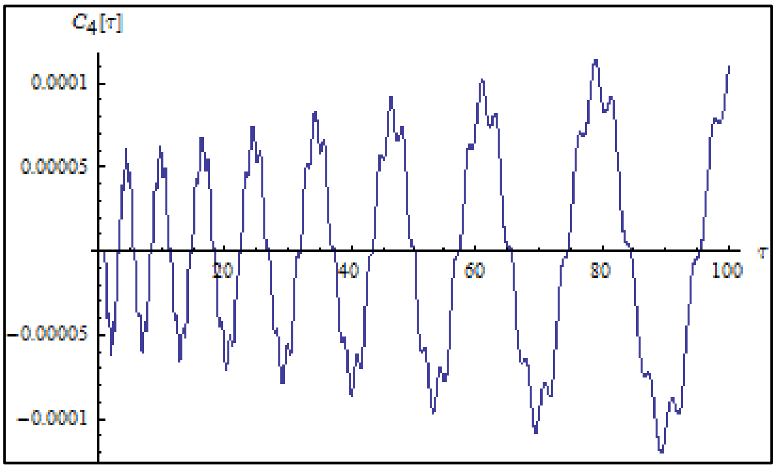

Fig. 4 Resonance at Fourth Mode

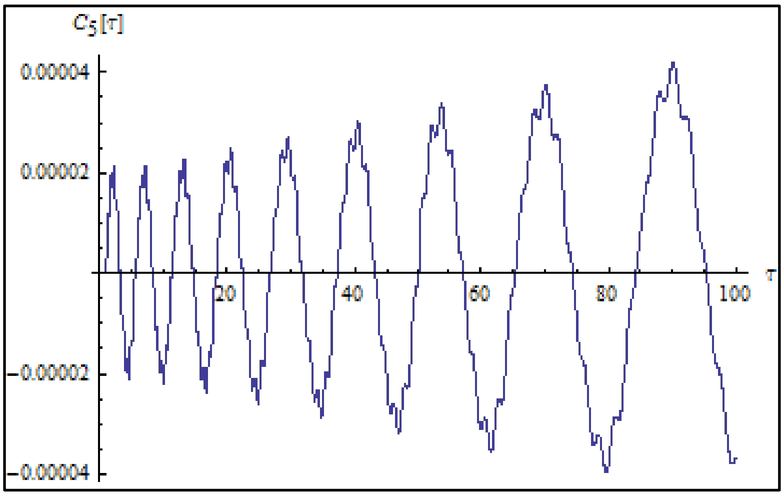

Fig. 5 Resonance at Fifth Mode

IV. DISCUSSION AND CONCLUSION

A longitudinally vibrating rod that is accreting in both the length and the cross-sectional area was considered. A partial differential equation modelling the dynamics of such a rod was derived. The differential equation was transformed into an infinite system of ordinary differential equations by the use of the Galerkin-Kantorovich method. This system of equations was conveniently truncated to a system of five equations. The solutions to this system of equations was obtained numerically using the *Mathematica*® 7.0 software.

The solutions exhibited a trend of increasing amplitude of vibration at any given mode of vibration. In this trend of exhibition it could also be seen that there is a decrease in the frequency of vibration. This behavioural pattern clearly indicating the resonance phenomenon. It could also be noticed from the graphical solutions that this pattern is also accompanied by an increase in the wavelength. It is again clearly noticeable that as we move from one mode of vibration to the next, there is a marked decrease in the amplitude of vibration.

It is the intention of the authors of this paper to extend this work into the next project, whereby for the similar longitudinally vibrating rod, we will use the Rayleigh-Love model, which is more accurate than the Classical model. The authors of this paper also intend investigating (the work that is already in progress) the behavioural patterns and dynamics of the similar rod when subjected to damped and forced vibrations, for both the Classical model and the Rayleigh-Love model.

V. ACKKNOWLEDGEMENT

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VI. REFERENCES

[1] M Shatalov, AV Manzhirov., &I Fedotov, On the resonant behaviour of longitudinally vibrating accreting rods. *SACAM2012*, 2012.

[2] M Shatalov, I Fedotov, G Lekalakala &J Bidie, Quasi-resonance behaviour of viscoelastic rods subjected to free and forced longitudinal vibrations. *IUTAM2015*. *Moscow, Russia*, 2015.

[3] AD Polyanin, AV Manzhirov, HandBook of Mathematics for Engineers and Scientists. Chapman & Hall/CRC, Boca Raton, Florida, USA, 2002.

[4] MLG Lekalakala, M Shatalov&I Fedotov, The Behavioural Pattern of the Longitudinally Vibrating Rod that is Accreting in Cross-Sectional Area. *IUTAM2015, Moscow, Russia* 2015.